March 19, 2013

MCS 452 MT1 Questions

1) Consider the metric space $(C[0,1], d)$ where

$$d(f,g) = \int_0^1 |f(x) - g(x)| dx$$

Let $f(x) = e^{-x}$ for $x \in [0,1]$.
i) Write the open ball $B_d(f,1)$ explicitly as a set,
ii) Which of the following functions in $C[0,1]$ are in $B_d(f,1)$?
a) g(x) = 0
b) g(x) = 1
Are any of these functions in $B_d(f,1/2)$? Justify your answers by explicit calculations.

2) a) Show that

$$d_2((x_n,y_n),(a,b)) \leq \sqrt{2}.d_\infty((x_n,y_n),(a,b))$$

with the usual metric d_2 and the maximum metric
$d_\infty((x_n,y_n),(a,b)) = \max\{|x_n - a|, |y_n - b|\}$.
b) Prove that a sequence $\{(x_n,y_n)\}$ converges to the limit $(a,b) \in \mathbb{R}^2$ using the usual metric d_2 if and only if it converges with respect to the metric $d_\infty = \max\{|x_n - a|, |y_n - b|\}$.

3) The Euclidean metric d on $\mathbb{R}^n$ is defined by

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots (x_n - y_n)^2}$$

for $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$.
a) Consider $(\mathbb{R}^3, d)$. Show that the sequence whose n^{th} term is (x_n, y_n, z_n) converges to (x,y,z) as $n \to \infty$ in the metric space $(\mathbb{R}^3, d)$ if and only if , $\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z$.
b) Find the limit in $(\mathbb{R}^3, d)$ of the sequence whose n^{th} term is

$$(\cos(\frac{3}{n+1}), \ln(\frac{n+5}{n}), \frac{3n+1}{n+2})$$

and calculate the distance between this limit and the origin.

4) On $C[0,1]$ define the operator T by

$$(Tx)(s) = s^2 \int_0^1 x(t)dt$$

where $x \in C[0,1]$ and $s \in [0,1]$. Show that T is linear and find $||T||$.